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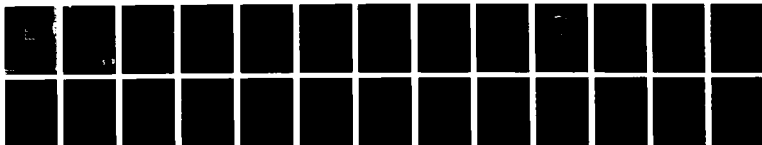
GENERATION OF SURFACE GRIDS THROUGH ELLIPTIC PARTIAL
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Generation of Surface Grids Through Elliptic Partial
Differential Equations for Aircraft and
Missile Configurations



by

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1. INTRODUCTION

The problem of numerical grid generation is of current interest in many branches of engineering particularly in aeronautics, mechanical, and civil engineering. The spatial grids are generated either by algebraic methods using various spline and transfinite interpolations or by solving certain partial differential equations. In this regard, a book [1], review articles [2], [3], and conference proceedings [4, 5, 6] may be consulted.

The progress report under consideration is in reference to the problem of grid generation in a given surface by solving a set elliptic partial differential equations. The mathematical model used in the present research has been developed by the author under previous AFOSR grants and has been described in publications [7 - 16]. Reference [16] has been attached as an appendix in which Eqs. (19) and (21) describe the mathematical model.

The developed mathematical model has been used to generate the Cartesian coordinates as functions of the curvilinear coordinates when the surface in which the coordinates are to be generated has been specified either analytically or by discrete data points. In most practical situations the surface is usually specified by discrete data, and therefore, there is a need to develop computer routines to fit a global equation of the form $F(x,y,z) = 0$. For complicated body shapes e.g. an airplane, we have to divide the surface into suitable sections and then fit the functions separately. The need for specifying the function F is due to the fact that the forcing term R in Eqs. (19) or (21) of the appendix is derived from F . In the period under consideration we have devised various routines for the above noted purpose and they are described in Section 2. The developed

computer code has been used in many complicated shaped surfaces. Figure 1 shows some current results.

2. NUMERICAL SCHEMES

A computer program for the numerical solution of Eq. (21) has been developed by using point and line SOR. The essential difference between the grid generation in a flat space and in a curved surface is in the appearance of the forcing term on the right of Eqs. (19) and (21). The term R in Eqs. (19) and (21) depends on the first and second partial derivatives of the fitted function F with respect to the Cartesian coordinates. (Refer to [15] for the formula of R in terms of F). The following routines have been developed and used in a variety of problems.

1. Multidimensional least square technique.
2. "Overlapping" least square technique.
3. Fourier decomposition of each section of a surface and then blending them to obtain the equation of a square.

Beside the development of the above techniques, we have also developed the following two separate programs in the solution of Eq. (21).

1. For the acceleration of the iterative process, a "multigrid" technique in the solution of Eq. (21) has been incorporated. From the test cases conducted so far it looks that the solution time with multigrid is not significantly lower than that of SOR. Research in this area is, however, continuing.
2. A routine for the calculation of the optimum acceleration parameter has been developed which works quite well with the SOR technique.

3. CONCLUSIONS

Computer programs have been developed which solve the problem of numerical coordinate generation in a given surface. The mathematical model is formed of a set of elliptic equations and thus the resulting coordinates in the surface are sufficiently smooth. The grid generation in arbitrary surface is accomplished by dividing the surface into suitable subregions and then joining them to obtain the coordinates in the integrated surface.

REFERENCES

1. THOMPSON, J. F., WARSI, Z.U.A., and MASTIN, C.W., Numerical Grid Generation: Foundations and Applications, North-Holland, Amsterdam, 1985.
2. THOMPSON, J.F., WARSI, Z.U.A., and MASTIN, C.W., 'Boundary-Fitted Coordinate Systems for Numerical Solution of Partial Differential Equations', Journal of Computational Physics, 1982, 47, pp. 1-108.
3. EISEMAN, P.R., 'Grid Generation for Fluid Mechanics Computations', Ann. Rev. Fluid Mech., 1985, 17, pp. 487-522.
4. THOMPSON, J.F. (Ed.), Numerical Grid Generation, North-Holland, Amsterdam, 1982.
5. GHIA, K.N., and GHIA, U., (Eds.), Advances in Grid Generation, ASME Pub. No. FED-Vol. 5, 1983.
6. HAUSER, J., and TAYLOR, C., (Eds.), Numerical Grid Generation in Computational Fluid Dynamics, Pineridge Press, Ltd., (1986).
7. WARSI, Z.U.A., 'A Method for the Generation of General Three-Dimensional coordinates Between Bodies of Arbitrary Shapes', Mississippi State University, Engineering and Industrial Research Station, Report MSSU-EIRS-80-7, 1980.
8. WARSI, Z.U.A., 'Tensors and Differential Geometry Applied to Analytic and Numerical Coordinate Generation', Ibid, MSSU-EIRS-81-1, 1981.
9. WARSI, Z.U.A., 'Basic Differential Models for Coordinate Generation', in Numerical Grid Generation, Ed. Thompson, J. F., 1982, pp. 41-77.
10. WARSI, Z.U.A., and ZIEBARTH, J.P., 'Numerical Generation of Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes', in Numerical Grid Generation, Ed. Thompson, J.F., 1982, pp. 717-728.
11. WARSI, Z.U.A., 'A note on the Mathematical Formulation of the Problem of Numerical Coordinate Generation', Quart. Appl. Math., 1983, 41, pp. 221-236.
12. WARSI, Z.U.A., 'Generation of Three-Dimensional Grids Through Elliptic Differential Equations', VonKarman Institute Lecture Series on Computational Fluid Dynamics, 1984-04
13. WARSI, Z.U.A., and TIARN, W.N., 'Numerical Grid Generation Through Second Order Differential Geometric Models', 11th IMACS World Congress, Oslo, Norway, August 1985.
14. WARSI, Z.U.A., and TIARN, W.N., 'Surface Mesh Generation Using Elliptic Equations', in Numerical Grid Generation in Computational Fluid Dynamics, Eds. Hauser, J., and Taylor, C., (1986), pp. 95-110.

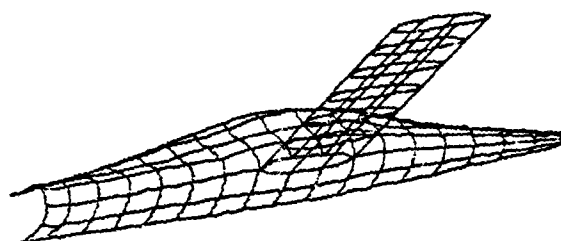
15. WARSI, Z.U.A., 'Numerical Grid Generation in Arbitrary Surfaces Through A Second-Order Differential-geometric Model', Journal of Computational Physics, 1986, 64, pp. 82-96.
16. WARSI, Z.U.A., 'A Synopsis of Elliptic PDE Models for Grid Generation', Appl. Math. and Comput., 1987, 21, pp. 295-311.

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2. "A Synopsis of Elliptic PDE Models for Grid Generation", by Z.U.A. Warsi, Applied Math. and Comp., Vol. 21, (1987), pp. 295-311.
3. "Generation of Surface coordinates by Elliptic Partial Differential Equations", by Z.U.A. Warsi, NASA Conf. Publ. 2379, June (1985).
4. "Numerical Grid Generation Through Second Order Differential-Geometric Models", by Z.U.A. Warsi, and W.N. Tiarn, 11th IMACS World Congress, Oslo, Aug. (1985).
5. "Generation of Three-Dimensional Coordinates by Using the Method of Parallel Surface", by J.H. Lee, MS Thesis, Mississippi State University, Dec. (1985).
6. "Surface Mesh Generation Using Elliptic Equations", by Z.U.A. Warsi, and W.N. Tiarn, article in Numerical Grid Generation In Computational Fluid Dynamics, Eds. J. Hauser, and C. Taylor, Pineridge Press, Ltd. (1986), pp. 95-110.



a



b

Figure 1. (a) Generation of surface grids on the fuselage of an airplane with wing-body intersection shown.
 (b) Integration of wing and fuselage generated coordinates to exhibit the coordinates on wind-body combination.

APPENDIX

A Synopsis of Elliptic PDE Models for Grid Generation*

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ABSTRACT

This paper is devoted to an analytical comparison of the various elliptic partial-differential-equation (PDE) models which are in current use for grid generation. These comparisons, particularly between the equations from the Laplace-Poisson system and the equations from a Gaussian approach, have yielded useful expressions connecting the 3D Laplacians and the surface Beltramians. This effort has specifically been successful when the transverse coordinate leaving the surface is orthogonal to the surface. Equations which are derivable from Cartesian-type Poisson equations and those obtained by using the variational principle in surface coordinates have also been considered.

1. INTRODUCTION

The problem of generating spatial coordinates by numerical methods is of much importance in many branches of engineering mechanics. A review of various methods of coordinate generation in both two- and three-dimensional Euclidean spaces is available in Thompson, Warsi, and Mastin [1, 2].

This paper is exclusively directed to a collection and analytical comparisons of the various elliptic partial-differential-equation (PDE) models which are currently in use for numerical coordinate generation.

The theory of grid generation does not depend on any set of so-called conservation laws, and thus a variety of equations and methods of different characters can be used to obtain the grids. Any consistent method, depending either on the solution of PDEs or any algebraic method, can be used to obtain intersecting trajectories in either 2D or 3D Euclidean space.¹

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¹The question of space comes into the picture when it is realized that all the metric coefficients g_{ij} cannot be selected arbitrarily. In fact, these metric coefficients must be so selected that a set of second-order PDEs become satisfied. For details refer to Warsi [3].

In light of the above discussion, any consistent set of PDEs is sufficient to form a mathematical model. In bounded domains the most natural choice is that of a mathematical model formed of elliptic PDEs. The simplest set of equations, which also yields the smoothest grid, consists of the Laplace equations of the curvilinear coordinates in the Cartesian physical space. As a next logical step, a set of Poisson equations can be selected so as to have a degree of control of the distribution of grid lines. On inverting these equations a set of quasilinear PDEs are obtained [see Equations (2), (4)].

After the development of the problem of grid generation through solving the inverted forms of the Laplace and Poisson equations in 2D Euclidean domains by Allen [4], Winslow [5], Chu [6], and Thompson et al. [7], a logical extension was to use the same equations for 3D domains, as has been done by Thompson and Mastin [8]. Parallel to the above-noted developments, Warsi [3, 9, 10] proposed a Gaussian approach which basically generates surfaces and thus can be used either for generating curvilinear coordinates in a given surface [11] or for generating 3D coordinates by generating a series of surfaces starting from the data on the given surfaces [9, 12]. The Gaussian approach in fact depends on a manipulation of the formulae of Gauss for a surface, and thus the resulting equations have the surface coordinates as the independent variables. This manipulation introduces the Beltramians of the curvilinear coordinates and the sum of the principal curvatures of the surface in a natural way. *Since the formulae of Gauss for a surface hold true for any allowable coordinate system introduced in the surface, the equations proposed by Warsi [3, 9, 10] must also have the same properties. Further, because of the use of Gauss formulae, the proposed equations are optimal in the sense that the number of terms and the amount of information in the equations is just sufficient for the generation of either surfaces or coordinates.*

In this regard we can justifiably call a mathematical model "optimal" if it can be reduced to the form of the proposed equations [cf. Equation (19a)].

Beside the Laplace-Poisson system and the Gaussian approach, we have also derived the surface generating equations by the use of the variational principle. The resulting equations are near optimal for the Gaussian system.

2. BASIC ELLIPTIC MODELS

Poisson Equations as Grid Generators

Since the publication of the TTM method [7], there has been extensive use of the Poisson equations in the physical r -space to generate both 2D and 3D grids [2]. In practically all cases the main aim is to have those equations in which the computational coordinates appear as the independent variables,

and therefore, the Poisson equations have to be inverted by making the physical coordinates $r = (x, y, z)$ the dependent variables. This inversion can of course be carried out in a nontensorial manner by using the chain rule of partial differentiation. However, it is much simpler to follow either of the following two methods to attain the same result.

Method I: Inner multiplication of Equation (A2)² by g^{ij} and the use of Equation (A4) results in

$$g^{ij}r_{,ij} + (\nabla^2 x^k)r_{,k} = 0. \quad (1)$$

Introducing the second-order differential operator

$$D = gg^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$$

into Equation (1), we get

$$Dr + g(\nabla^2 x^k)r_{,k} = 0, \quad (2)$$

which is the desired equation in vector form in the computational plane.

Method II: In Equation (A3), writing $\phi = r = (x, y, z)$ and again using Equation (A4), we obtain the same equation (2).

To form a closed system of equations from Equation (2) one has to specify the Laplacians $\nabla^2 x^k$ in a suitable fashion. As described in Reference [3], a general specification for the Laplacians is to write

$$\nabla^2 x^k = g^{ij}P_{ij}^k, \quad k = 1, 2 \text{ or } k = 1, 2, 3, \quad (3)$$

and then Equation (1) takes the form

$$g^{ij}(r_{,ij} + P_{ij}^k r_{,k}) = 0. \quad (4)$$

In Equation (3), the $P_{ij}^k = P_{ji}^k$ are intended to be arbitrarily specified control functions. Equating the right-hand sides of Equation (A4) and Equation (3),

²Refer to the Appendix.

we have

$$g^{ij}P_{ij}^k = -g^{ij}\Gamma_{ij}^k. \quad (5)$$

Since g^{ij} are not arbitrary, we conclude that

$$P_{ij}^k = -\Gamma_{ij}^k$$

for all values of i, j, k .

Recursive Property of P_{ij}^k

We now impose the following requirement on P_{ij}^k : If the coordinates are such that their Laplacians vanish, i.e., $\nabla^2 x^k = 0$, then the control functions P_{ij}^k vanish individually for all indices i, j, k .

The importance of this requirement becomes clear by considering Equation (5) in which Γ_{ij}^k are the Christoffel symbols. Thus, when the Laplacians of the coordinates vanish, the right-hand side of Equation (5) vanishes as an inner sum, but the left-hand side vanishes due to the imposed restriction of "individual vanishing" of P_{ij}^k for such coordinates. Denoting the coordinates which satisfy the Laplace equations by $x_{(0)}^i$, we then have

$$P_{ij(0)}^k = 0, \quad \Gamma_{ij(0)}^k \neq 0, \quad g_{(0)}^{ij}\Gamma_{ij(0)}^k = 0. \quad (6)$$

We now consider those coordinates which satisfy the Poisson equations. In these coordinates it is of interest to know the relation between the successive P 's under successive coordinate transformations. To this end, we consider two successive allowable transformations denoted by $x_{(m-1)}^i$ and $x_{(m)}^i$, with $x_{(0)}^i$ as those coordinates which satisfy the Laplace equation. Thus, in the $x_{(m-1)}^i$ coordinates, Equation (4) is

$$g_{(m-1)}^{ij} \left(\frac{\partial^2 r}{\partial x_{(m-1)}^i \partial x_{(m-1)}^j} + P_{ij(m-1)}^p \frac{\partial r}{\partial x_{(m-1)}^p} \right) = 0, \quad (7)$$

and in the $x_{(m)}^i$ coordinates

$$g_{(m)}^{ij} \left(\frac{\partial^2 r}{\partial x_{(m)}^i \partial x_{(m)}^j} + P_{ij(m)}^p \frac{\partial r}{\partial x_{(m)}^p} \right) = 0. \quad (8)$$

Using the transformation law (A7) in the form

$$g_{(m-1)}^{ij} = \frac{\partial x_{(m-1)}^i}{\partial x_{(m)}^p} \frac{\partial x_{(m-1)}^j}{\partial x_{(m)}^n} g_{(m)}^{pn}, \quad (9)$$

and the chain rule of partial differentiation of r in Equation (7), and comparing the resulting equation with Equation (8), we get

$$P_{ij(m)}^p = \left(P_{kn(m-1)}^s \frac{\partial x_{(m)}^p}{\partial x_{(m-1)}^s} + \frac{\partial^2 x_{(m)}^p}{\partial x_{(m-1)}^k \partial x_{(m-1)}^n} \right) \frac{\partial x_{(m-1)}^k}{\partial x_{(m)}^i} \frac{\partial x_{(m-1)}^n}{\partial x_{(m)}^j}, \quad (10)$$

establishing a relation between the successive P 's. This is how P_{ij}^k transform from one coordinate system to the other. Also, using Equation (A9) in Equation (10), we have

$$P_{ij(m)}^p = -\Gamma_{ij(m)}^p + (P_{kn(m-1)}^s + \Gamma_{kn(m-1)}^s) \frac{\partial x_{(m)}^p}{\partial x_{(m-1)}^s} \frac{\partial x_{(m-1)}^k}{\partial x_{(m)}^i} \frac{\partial x_{(m-1)}^n}{\partial x_{(m)}^j}. \quad (11)$$

Equation (11) establishes a relation between the P 's and Γ 's. In particular, for $m=1$, Equation (10) reduces to Equation (63) of Reference [3].

From the recursive relation (10) it is a straightforward matter to show that the transformation $x_{(0)}^i \rightarrow x_{(1)}^i \rightarrow x_{(2)}^i$ of P_{ij}^k is the same as the transformation $x_{(0)}^i \rightarrow x_{(2)}^i$. The use of the chain rule for

$$\frac{\partial x_{(2)}^i}{\partial x_{(1)}^j}, \quad \frac{\partial^2 x_{(2)}^i}{\partial x_{(1)}^j \partial x_{(1)}^k}$$

and of the formula

$$\frac{\partial^2 x_{(0)}^p}{\partial x_{(1)}^i \partial x_{(1)}^j} = - \frac{\partial^2 x_{(1)}^i}{\partial x_{(0)}^r \partial x_{(0)}^s} \frac{\partial x_{(0)}^p}{\partial x_{(1)}^r} \frac{\partial x_{(0)}^s}{\partial x_{(1)}^i} \frac{\partial x_{(0)}^t}{\partial x_{(1)}^j} \frac{\partial x_{(0)}^u}{\partial x_{(1)}^t}$$

gives the required result.

Other Poisson Systems

It is also possible to have equations in which the dependent variables are non-Cartesian, e.g., cylindrical or spherical coordinates. In a paper by Chia

et al. [13], the computational coordinates are assumed to satisfy a Poisson system in the cylindrical coordinates.

To generalize this concept, let x^s be a coordinate system on which the coordinate system \bar{x}^s (e.g., cylindrical) is to be generated. From Equation (A4),

$$\nabla^2 \bar{x}^s = -\bar{g}^{ij} \bar{\Gamma}_{ij}^s \quad (12)$$

and

$$\begin{aligned} \nabla^2 x^k &= -g^{ij} \Gamma_{ij}^k = g^{ij} P_{ij}^k \\ &= P^k \quad (\text{say}). \end{aligned} \quad (13)$$

In Equation (12) both the metric coefficients and the Christoffel symbols are already known in terms of the \bar{x}^s coordinate system. [If \bar{x}^s are the Cartesian coordinates, then Equation (12) is an identity.] From Equation (A3) we then have

$$g^{ij} \frac{\partial^2 \bar{x}^s}{\partial x^i \partial x^j} + P^k \frac{\partial \bar{x}^s}{\partial x^k} = -\bar{g}^{ij} \bar{\Gamma}_{ij}^s, \quad (14)$$

which are the transformed equations in the computational space. By using Equation (A10) we can also write Equation (14) as

$$\bar{g}^{mn} C_m^i C_n^j \frac{\partial^2 \bar{x}^s}{\partial x^i \partial x^j} + J^2 P^k \frac{\partial \bar{x}^s}{\partial x^k} = -(J)^2 \bar{g}^{ij} \bar{\Gamma}_{ij}^s, \quad (15)$$

where

$$J = \det \left(\frac{\partial \bar{x}^i}{\partial x^j} \right).$$

Recently Fujii [14] used a Cartesian-type Poisson system between two curvilinear coordinate systems. Let x^s and \bar{x}^s be two curvilinear systems. Consider the Cartesian-type Poisson system

$$\sum_p \frac{\partial^2 x^r}{\partial \bar{x}^p \partial \bar{x}^p} = P^r, \quad (16)$$

where x' is the curvilinear system on which the \bar{x}^p -system (e.g., spherical) is to be generated. To obtain the inversion of Equation (16), we take the inner product of Equation (A11) with $\partial \bar{x}' / \partial x'$ to obtain

$$\frac{\partial^2 \bar{x}'}{\partial x' \partial x^j} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^n} = - \frac{\partial^2 x'}{\partial \bar{x}^k \partial \bar{x}^n} \frac{\partial \bar{x}'}{\partial x^i}. \quad (17)$$

Setting $k = n = 1, k = n = 2, k = n = 3$ and adding, we get

$$C^{ij} \frac{\partial^2 \bar{x}'}{\partial x' \partial x^j} = - \frac{\partial \bar{x}'}{\partial x^i} \sum_p \frac{\partial^2 x'}{\partial \bar{x}^p \partial \bar{x}^p},$$

so that

$$C^{ij} \frac{\partial^2 \bar{x}'}{\partial x' \partial x^j} = - \frac{\partial \bar{x}'}{\partial x^i} P^i, \quad (18)$$

where

$$C^{ij} = \sum_p \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^p} = \frac{1}{J^2} \sum_p C_p^i C_p^j.$$

Equation (18) is the inverted form of Equation (16). Similarly to the case of actual Poisson system, the functions P^i are again the arbitrary specified functions. Alternatively, we can also write

$$P^i = C^{ij} P'_{ij},$$

and then P'_{ij} are the arbitrary control functions.

Beltramians as Grid Generators in Curved Surfaces

The problem of generating curvilinear coordinates in a surface has important applications in many branches of engineering. In this paper we are interested only in one differential model which generates coordinates in surfaces embedded in R_3 and which reduces to Equation (2) in a natural way when the surface degenerates into a plane. To achieve this aim, we consider the formulae of Gauss [Equation (A12)] for a surface $\nu = \text{const}$. Inner multiplication of Equation (A12) by $C_{,g}{}^{a\beta}$ yields

$$D\mathbf{r} + G_{,s}(\Delta^{(s)}_2 \mathbf{x}^s)_{\mathbf{r},s} = \mathbf{n}^{(r)} R, \quad (19a)$$

where

$$\begin{aligned} D &= G_\nu g^{\alpha\beta} \partial_{\alpha\beta}, \quad n^{(\nu)} = \text{unit surface normal}, \\ G_\nu &= g_{\alpha\alpha} g_{\beta\beta} - (g_{\alpha\beta})^2, \quad \nu, \alpha, \beta \text{ cyclic}, \\ R &= G_\nu g^{\alpha\beta} b_{\alpha\beta} = (k_I^{(\nu)} + k_{II}^{(\nu)}) G_\nu, \end{aligned} \quad (19b)$$

$b_{\alpha\beta}$ are the coefficients of the second fundamental form, and $k_I^{(\nu)}$, $k_{II}^{(\nu)}$ are the principal curvatures at a point of the surface $\nu = \text{const}$. The Beltramians $\Delta_2^{(\nu)} x^\delta$ are defined by Equation (A14). It is readily seen that when the surface $\nu = \text{const}$ degenerates into a plane, then $R = 0$ and Δ_2 becomes the Laplace operator.

Equation (19a) can be used either for the generation of coordinates in a given surface or for the generation of 3D spatial coordinates between two given surfaces. In the latter case it has been shown [10] that

$$R = G_\nu g^{\alpha\beta} \Gamma_{\alpha\beta}^{\nu'} \lambda^{(\nu')}, \quad \lambda^{(\nu')} = n^{(\nu)} \cdot r_{\nu'}, \quad (19c)$$

where $\Gamma_{\alpha\beta}^{\nu'}$ are the 3-space Christoffel symbols and $x^{\nu'}$ is the transverse coordinate.

Surface Coordinates

The surface-oriented generating system of equations, with the option of arbitrary coordinate control, is now obtained by putting suitable restrictions on the Beltramians appearing in Equation (19a). Similar to Equation (3), a general specification of the Beltramians is

$$\Delta_2^{(\nu)} x^\delta = g^{\alpha\beta} P_{\alpha\beta}^\delta, \quad (20)$$

where $P_{\alpha\beta}^\delta$ satisfy all the properties stated earlier, including Equations (10) and (11) with Γ replaced by T .

To be specific, we take the surface $x^3 = \text{const}$ as the given surface and $x^1 = \xi$, $x^2 = \eta$. Then Equation (19a) becomes

$$Lr = n^{(3)} R, \quad (21a)$$

where

$$\begin{aligned} L &= g_{22} \partial_{\xi\xi} - 2g_{12} \partial_{\xi\eta} + g_{11} \partial_{\eta\eta} + \bar{P} \partial_\xi + \bar{Q} \partial_\eta, \\ \bar{P} &= g_{22} P_{11}^1 - 2g_{12} P_{12}^1 + g_{11} P_{22}^1, \\ \bar{Q} &= g_{22} P_{11}^2 - 2g_{12} P_{12}^2 + g_{11} P_{22}^2. \end{aligned} \quad (21b)$$

3. SURFACE-GRID EQUATIONS FROM THE LAPLACIANS

In this section an attempt is made to establish a relation between the surface equations derived by using the formulae of Gauss [Equation (19a)] and the surface equations as derived from the inversions of the 3D Laplacians.

Let ξ, η, ζ be a general curvilinear coordinate system in R^3 such that ξ, η form the coordinates in the surface $\zeta = \text{const}$, with ζ as the transverse coordinate. Starting from Equation (B.3) of Reference [1] and using Equation (A2), we get (refer to [11] for more details)

$$\Omega r = \frac{G_3(k_I^{(3)} + k_{II}^{(3)})r_I}{\lambda}, \quad (22a)$$

where the differential operator Ω is

$$\Omega = D + T^\alpha \frac{\partial}{\partial x^\alpha} \quad (22b)$$

$$= D + (S^\alpha + G_3 \Delta_2 x^\alpha) \frac{\partial}{\partial x^\alpha}. \quad (22c)$$

In (22b, c) the differential operator D defined in (19b) is

$$D = g_{22} \partial_{\xi\xi} - 2g_{12} \partial_{\xi\eta} + g_{11} \partial_{\eta\eta}, \quad (22d)$$

and

$$T^\alpha = 2g_{12} \Gamma_{12}^\alpha - g_{22} \Gamma_{11}^\alpha - g_{11} \Gamma_{22}^\alpha, \quad (22e)$$

$$S^\alpha = 2g_{12}(\Gamma_{12}^\alpha - T_{12}^\alpha) - g_{22}(\Gamma_{11}^\alpha - T_{11}^\alpha) - g_{11}(\Gamma_{22}^\alpha - T_{22}^\alpha), \quad (22f)$$

where $\alpha = 1, 2$. Further

$$\lambda = n \cdot r_I, \quad (22g)$$

n being the unit surface normal vector on $\zeta = \text{const}$.

It must be emphasized that Γ_{jk}^i and T_{ab}^α are the 3-space and 2-space Christoffel symbols of the second kind respectively, and in general they are not equal to each other at $\zeta = \text{const}$. Thus, a comparison of Equations (19a) and (22a) shows that the two sets of surface generators are entirely different. However, it has been shown below that Equations (19a) and (22a) become

exactly the same equations under the following conditions:

- (i) When the surface degenerates into a plane, in which case ξ, η are the curvilinear coordinates (of any nature) in a plane.
- (ii) When ξ, η are any general coordinates in a surface, but ζ (the transverse coordinate) is orthogonal to the surface.

Case (i) is patently straightforward. For case (ii), noting that ζ is orthogonal to the surface, we have

$$g_{13} = g_{23} = 0, \quad (23a)$$

and

$$n = \frac{r_\zeta}{\sqrt{g_{33}}}, \quad \lambda = \sqrt{g_{33}}. \quad (23b)$$

Under the conditions (23a) it is easy to show that

$$\Gamma_{\beta\gamma}^\alpha = T_{\beta\gamma}^\alpha,$$

where all Greek indices assume only two values (here 1, 2). Thus $S^\alpha = 0$, and using (23b) we find that Equations (19a) and (23a) become the same equations under the condition of orthogonality of the ζ -coordinate to the surface $\zeta = \text{const}$. Further, under this condition the Beltramians and the Laplacians of ξ and η are related through the following equations:

$$\nabla^2 \xi = \Delta_2 \xi - \frac{\Gamma_{33}^1}{g_{33}}, \quad (24a)$$

$$\nabla^2 \eta = \Delta_2 \eta - \frac{\Gamma_{33}^2}{g_{33}}. \quad (24b)$$

Further,

$$\nabla^2 \zeta = -\frac{\Gamma_{33}^3}{g_{33}} - \frac{k_1^{(3)} + k_{11}^{(3)}}{\sqrt{g_{33}}}, \quad (24c)$$

where

$$k_1^{(3)} + k_{11}^{(3)} = -\frac{1}{2\sqrt{g_{33}}} \frac{\partial G_3}{\partial \zeta}, \quad (24d)$$

and [referring to (19b)]

$$G_3 = g_{11}g_{22} - (g_{12})^2.$$

4 SURFACE GENERATING EQUATIONS BASED ON A VARIATIONAL PRINCIPLE

The use of a variational principle in the generation of both 2D and 3D grids has already been considered by Brackbill and Saltzman [15, 16] and Thompson and Warsi [17]. In this section we shall consider only the surface grid generation problem based on the use of a variational principle. In essence the following analysis is a unified approach to both the plane 2D and surface 2D cases from a variational viewpoint.

Let x^1, x^2 be the coordinates in a surface. Consider the surface functional

$$I = \int \sqrt{G_3} \phi \, dx^1 \, dx^2, \quad (25)$$

where $G_3 = g_{11}g_{22} - (g_{12})^2$ and ϕ is a specified function. The condition $\delta I = 0$ then leads one to the Euler-Lagrange equations (using the summation convention)

$$\frac{\partial}{\partial x_r} (\sqrt{G_3} \phi) - \frac{\partial}{\partial x^\beta} \frac{\partial (\sqrt{G_3} \phi)}{\partial x_{r,\beta}} = 0, \quad (26)$$

where x_r ($r = 1, 2, 3$) are the rectangular Cartesian coordinates, x^β are the curvilinear surface coordinates, and

$$x_{r,\beta} = \frac{\partial x_r}{\partial x^\beta}, \quad x_{r,\alpha\beta} = \frac{\partial^2 x_r}{\partial x^\alpha \partial x^\beta}.$$

From Equation (26), it is a purely algebraic problem to show that

$$L x_r = \frac{\sqrt{G_3}}{2} \frac{\partial}{\partial x^r} \left(\frac{1}{\sqrt{G_3}} \frac{\partial G_3}{\partial x_{r,r}} \right), \quad (27a)$$

where

$$L = G_3 \left[g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + (\Delta_2 x^\alpha) \frac{\partial}{\partial x^\alpha} \right]. \quad (27b)$$

Let now ϕ be a function of $x_{r,\beta}$. Then expanding Equation (26) and using Equation (27), we get

$$Lx_r = -M, \quad (28a)$$

where

$$M = \frac{\phi}{2} \left(\frac{\partial G_3}{\partial x_{r,\beta}} \frac{\partial \phi}{\partial x^\beta} + \frac{\partial \phi}{\partial x_{r,\beta}} \frac{\partial G_3}{\partial x^\beta} \right) + \frac{G_3}{\phi} \frac{\partial}{\partial x^\beta} \left(\frac{\partial \phi}{\partial x_{r,\beta}} \right). \quad (28b)$$

On the other hand, if ϕ is taken as

$$\phi = F/G_3, \quad (29a)$$

where F is still a function of $x_{r,\beta}$, then in place of (28b), we have

$$M = \frac{1}{2F} \left(\frac{\partial G_3}{\partial x_{r,\beta}} \frac{\partial F}{\partial x^\beta} + \frac{\partial G_3}{\partial x^\beta} \frac{\partial F}{\partial x_{r,\beta}} \right) - \frac{1}{2G_3} \frac{\partial G_3}{\partial x_{r,\beta}} \frac{\partial G_3}{\partial x^\beta} - \frac{G_3}{F} \frac{\partial}{\partial x^\beta} \left(\frac{\partial F}{\partial x_{r,\beta}} \right). \quad (29b)$$

The generating system (28a) with L defined in (27b) is similar to Equation (19a). However, the selection of the form of the function ϕ or F which yields the right-hand side of Equation (19a) seems to be a difficult task. One simple case is when $\phi = 1$. In this case the minimization of I implies [from (28a)]

$$Lx_r = 0,$$

and these are the equations for a minimal surface. The form (29a) is of interest because the choice

$$F = g_{11} + g_{22}, \quad (30a)$$

or

$$\phi = g^{11} + g^{22}, \quad (30b)$$

is equivalent to the "smoothness" problem in 2D plane coordinates, as has

been shown in Reference [15]. It must, however, be stated that "smoothness" of coordinates in a 2D plane problem is due to the satisfaction of the Laplace equation. No such criterion is obvious on using either of the equations (30) in (28a).

5. ANALYTICAL COMPARISONS AND CONCLUSIONS

Based on the foregoing analysis, we conclude as follows:

(a) The Laplace-Poisson system for 2D regions is optimal, since its inversion coincides exactly with the Gaussian equations in a plane. [Note: In a plane the right-hand side of Equation (19a) is zero and the Beltramians become the Laplacians.]

(b) The inversion of the Laplace-Poisson system for 3D regions for a constant coordinate value, viz., for a surface, does not reduce to the Gaussian equations (cf. Section 3) except when the transverse coordinate is taken as orthogonal to the surface [cf. Equation (22a)]. This implies that the Laplace-Poisson system in 3D regions with three nonorthogonal coordinates is not optimal, though it is a valid system. The extra terms (22f) should somehow be managed, and in practice, they are taken as part of the arbitrary specified control functions. This means that the generated coordinates will assume a distribution which may not be to one's desire. It must, however, be again emphasized that the terms (22f) vanish when the transverse coordinate is orthogonal to the surface. (In Reference [18], the author had to make other assumptions besides orthogonality.)

(c) For the elliptic system described by Equation (15) the conclusions discussed in (b) hold good.

(d) The generating system described by Equation (16) is a Cartesian-type Poisson system. It looks difficult to assess its optimality in relation to the Gaussian equations.

(e) The equations derived from the variational principle, viz., (28), are nearly optimal, though it looks difficult to find the appropriate function ϕ which makes the right hand side of (28a) the same as that of Equation (19a).

APPENDIX

The following formulae have been used in the main text of the paper and can be found in any standard text on tensors, e.g., [19], [20].

In an Euclidean space R^n (though here we are concerned only with R^2 or R^3), endowed with the curvilinear coordinates x^i , the first partial derivatives

of the covariant basis vectors a_i are expressible as linear functions of a_k :

$$\frac{\partial a_i}{\partial x^j} = \Gamma_{ij}^k a_k, \quad (A1)$$

where Γ_{ij}^k are the Christoffel symbols of the second kind. Here, and in what follows, a repeated lower and upper index will always imply summation over the range of index values. Equation (A1) can alternatively be written as

$$r_{,ij} = \Gamma_{ij}^k r_{,k}, \quad (A2)$$

where a comma preceding an index implies a partial derivative, and r is the position vector, viz.,

$$r = (x, y, z) \text{ or } (x_1, x_2, x_3).$$

The Laplacian of a scalar ϕ is given by

$$\nabla^2 \phi = g^{ij} (\phi_{,ij} - \Gamma_{ij}^k \phi_{,k}). \quad (A3)$$

From (A3), we have

$$\nabla^2 x^k = -g^{ij} \Gamma_{ij}^k \quad (A4)$$

The quantities g^{ij} and g_{ij} are respectively the contravariant and covariant components of the metric tensor, and the two are related as

$$g^{ij} = \frac{g_{pn} g_{rs} - g_{ps} g_{nr}}{g},$$

where (i, p, r) and (j, n, s) are to be taken in the cyclic permutations of $(1, 2, 3)$, and

$$g = \det(g_{ij}).$$

Let x' and \bar{x}' be two allowable coordinate systems in a Euclidean space such that each of the functions $\bar{x}' = \phi'(x')$ and $x' = \psi'(\bar{x}')$ define a transformation with

$$J = \det\left(\frac{\partial \bar{x}'}{\partial x'}\right) \neq 0, \quad \bar{J} = \det\left(\frac{\partial x'}{\partial \bar{x}'}\right) \neq 0.$$

On transformation of coordinates from x^i to \bar{x}^i , we have

$$\bar{g} = \frac{g}{J^2}, \quad g = \frac{\bar{g}}{J^2}, \quad (\text{A5})$$

$$\bar{g}_{pn} = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^n} g_{ij}, \quad (\text{A6})$$

$$\bar{g}^{pn} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^n}{\partial x^j} g^{ij}, \quad (\text{A7})$$

$$\bar{\Gamma}_{kn}^p = \Gamma_{ij}^p \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^n} + \frac{\partial^2 x^i}{\partial \bar{x}^k \partial \bar{x}^n} \frac{\partial \bar{x}^p}{\partial x^i}, \quad (\text{A8})$$

$$\frac{\partial^2 \bar{x}^p}{\partial x^k \partial x^n} = \Gamma_{kn}^p \frac{\partial \bar{x}^p}{\partial x^i} - \bar{\Gamma}_{ni}^p \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^p}{\partial x^n}. \quad (\text{A9})$$

The first partial derivatives of x^i with respect to \bar{x}^i are given by

$$\frac{\partial x^i}{\partial \bar{x}^j} = \frac{C_j^i}{J},$$

where

$$C_j^i = \frac{\partial \bar{x}^r}{\partial x^j} \frac{\partial \bar{x}^k}{\partial x^n} - \frac{\partial \bar{x}^r}{\partial x^n} \frac{\partial \bar{x}^k}{\partial x^j}, \quad (\text{A10})$$

and (i, r, n) and (j, r, k) are cyclic permutations of $(1, 2, 3)$.

The second partial derivatives of one set of coordinates are related with those of the other set as

$$\frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^n} = - \frac{\partial^2 \bar{x}^s}{\partial x^i \partial x^j} \frac{\partial x^r}{\partial \bar{x}^s} \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^n}. \quad (\text{A11})$$

We now consider a surface embedded in R^3 . All tensor indices associated with a surface will be denoted by Greek letters (except the letter ν). In contrast to Equation (A2), the formula of Gauss is

$$r_{,\alpha\beta} = T_{\alpha\beta}^\delta r_{,\delta} + n^{(\nu)} b_{\alpha\beta}, \quad (\text{A12})$$

where $T_{\alpha\beta}^{\delta}$ are the surface Christoffel symbols of the second kind, $b_{\alpha\beta}$ are the coefficients of the second fundamental form, and $n^{(\nu)}$ is the unit surface normal on the surface $\nu = \text{const}$. The values of ν and other Greek indices follow the following scheme:

- $\nu = 1$: Greek indices α, β , etc. assume values 2 and 3.
- $\nu = 2$: Greek indices α, β , etc. assume values 3 and 1.
- $\nu = 3$: Greek indices α, β , etc. assume values 1 and 2.

The Beltramian of a scalar ϕ is given by

$$\Delta_2^{(\nu)}\phi = g^{\alpha\beta}(\phi_{,\alpha\beta} - T_{\alpha\beta}^{\delta}\phi_{,\delta}). \quad (\text{A13})$$

From (A13), we have

$$\Delta_2^{(\nu)}x^{\delta} = -g^{\alpha\beta}T_{\alpha\beta}^{\delta}. \quad (\text{A14})$$

For a surface the formulae (A5)–(A9) are equally applicable with proper choice of indices and replacing Γ by T .

REFERENCES

- 1 J. F. Thompson, Z. U. A. Warsi, and C. W. Mastin, Boundary-fitted coordinate systems for numerical solution of partial differential equations—a review, *J. Comput. Phys.* 47:1–108 (1982).
- 2 J. F. Thompson, Z. U. A. Warsi, and C. W. Mastin, *Numerical Grid Generation: Foundations and Applications*, North-Holland, 1985.
- 3 Z. U. A. Warsi, Basic differential models for coordinate generation, in *Numerical Grid Generation* (J. F. Thompson, Ed.), North-Holland, 1982, pp. 41–77.
- 4 D. N. de G. Allen, Relaxation methods applied to conformed transformation, *Quart. J. Mech. Appl. Math.* 15:35–42 (1962).
- 5 A. J. Winslow, Numerical solution of the quasi-linear Poisson equation in a non-uniform triangular mesh, *J. Comput. Phys.* 2:149–172 (1966).
- 6 W. H. Chu, Development of a general finite difference approximation for a general domain, part I: Machine transformation, *J. Comput. Phys.* 8:392–408 (1971).
- 7 J. F. Thompson, F. C. Thames, and C. W. Mastin, Automatic numerical generation of body-fitted coordinate system for field containing any number of arbitrary two-dimensional bodies, *J. Comput. Phys.* 15:299–319 (1974).
- 8 C. W. Mastin and J. F. Thompson, Three-dimensional body-fitted coordinate systems for numerical solution of the Navier-Stokes equations, AIAA 78–1147.
- 9 Z. U. A. Warsi, A method for the generation of general three-dimensional coordinates between bodies of arbitrary shapes, Report No. MSSU-EIRS-80-7, Engineering and Industrial Research Station, Mississippi State Univ., 1980.

- 10 Z. U. A. Warsi, A note on the mathematical formulation of the problem of numerical coordinate generation, *Quart. Appl. Math.* 41:221-236 (1983).
- 11 Z. U. A. Warsi, Numerical grid generation in arbitrary surfaces through a second-order differential-geometric model, *J. Comput. Phys.*, to appear.
- 12 Z. U. A. Warsi, Generation of three-dimensional grids through elliptic differential equations, Computational Fluid Dynamics Lecture Series 1984-04, Von Karman Institute for Fluid Dynamics, Belgium, Mar. 1984.
- 13 U. Ghia and K. N. Chia, Numerical generation of a system of curvilinear coordinates for turbine cascade flow analysis, Report No. AFL 75-4-17, Dept. of Aerospace Engineering, Univ. of Cincinnati, Apr. 1975.
- 14 K. Fujii, Solution of Poisson equations for three-dimensional grid generation, NASA Tech. Mem. 85857, 1983.
- 15 J. Saltzman and J. U. Brackbill, Application and generalizations of variational methods for generating adaptive grids, in *Numerical Grid Generation* (J. F. Thompson, Ed.), North-Holland, 1982.
- 16 J. U. Brackbill, Coordinate system control: Adaptive meshes, in *Numerical Grid Generation* (J. F. Thompson, Ed.), North-Holland, 1982.
- 17 J. F. Thompson and Z. U. A. Warsi, Three-dimensional grid generation from elliptic systems, AIAA Paper No. 83-1905, 1983.
- 18 P. D. Thomas, Construction of composite three dimensional grids from subregion grids generated by elliptic systems, AIAA Paper No. 81-0996.
- 19 B. Spain, *Tensor Calculus*, Oliver and Boyd, 1953.
- 20 Z. U. A. Warsi, Tensors and differential geometry applied to analytic and numerical coordinate generation, Report No. MSSU-EIRS-81-11-1, Engineering and Industrial Research Station, Mississippi State Univ., 1981.

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